Case III : Equal roots.

Def^n of e^{At}

— A is an $n \times n$ matrix

$$e^{At} \equiv I_n + \frac{At}{1!} + \frac{A^2t^2}{2!} + \dots + \frac{A^n t^n}{n!} + \dots$$

Properties of e^{At}

i) e^{At} is an $n \times n$ matrix

2) $\frac{d}{dt}(e^{At}) = A e^{At}$

3) $e^{I\lambda t} = I e^{\lambda t}$.

Proof: 3

$$e^{I\lambda t} = I + \frac{I\lambda t}{1!} + \frac{(I\lambda)^2 t^2}{2!} + \dots$$

$$= I\left[1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \dots\right]$$

$$= I e^{\lambda t}.$$

Proof 2

$$\frac{d}{dt}(e^{At}) = A + A^2 t + \dots + \frac{A^{n+1} t^n}{n!} + \dots$$

$$= A\left[1 + \frac{At}{1!} + \dots + \frac{A^n t^n}{n!} + \dots\right]$$

$$= A e^{At}$$

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We seek the solution in the form
$x = e^{At} v$
where, v is $n \times 1$ matrix of constants.

Let v be the eigenvector corresponding to eigenvalue λ
$$[A - \lambda I] v = 0$$

We have,
$$x = e^{At} v$$
$$= e^{\lambda I t} e^{At} e^{-\lambda I t} v$$
$$= I e^{\lambda t} e^{(A-\lambda I)t} v \quad | \text{by ③ prop of } e^{At}.$$

$$= e^{\lambda t} \left[I + \frac{(A-\lambda I)t}{1!} + \frac{(A-\lambda I)^2 t^2}{2!} \cdots \right] v$$

If there exist a v, such that
$$[A - \lambda I) v \neq 0$$
$$[A - \lambda I]^2 v = 0$$
(A)

$\therefore$ The solution corresponds to v, is
$$e^{\lambda t} \left[I + \frac{(A-\lambda I)t}{1!} \right] v$$
(B)

Q). Find three linearly independent solutions of the differential eq^n

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x$$

Solution:

Eigenvalues of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\{(1-\lambda)(2-\lambda) - 0\} = 0$$

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, 1, 2$$

Hence $\lambda = 1$ is an eigenvalue of A with multiplicity two and $\lambda = 2$ is an eigenvalue of A with multiplicity one.

Eigenvector corresponding to corresponding eigenvalue $\lambda = 1$

$$[A - \lambda I] v = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $v_2 = 0$, and $v_3 = 0$

v_1 is arbitrary.

$$\therefore \quad x^1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is one solution of given diff. eq^n.

For the next solution

we find v such that

$$\left.\begin{array}{l} (A - \lambda I)^2 v = 0 \\ \text{but } (A - \lambda I) v \neq 0 \end{array}\right\} \text{from } A$$

i.e. $(A - I)^2 v = 0$ as $\lambda = 1$

but $\cancel{A - I)}$ $(A - I) v \neq 0$

Now,

$$(A-I)^2 v = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_3 = 0$$

and v_1, v_2 are arbitrary

Also $(A-I)v \neq 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq 0$$

$$\Rightarrow v_2 \neq 0 \text{ and } v_3 \neq 0.$$

$\therefore$ we take, $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

[we could choose any $v = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ for which $v_2 \neq 0$]

$$\therefore \quad x^2(t) = \cancel{e^{At}}\, e^{\lambda t}\left[I + \frac{(A-\lambda I)t}{1!}\right] v$$

from (B)

$$= e^t\left[I + (A-I)t\right] v$$

$$= e^t\left\{I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} t\right\}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= e^t\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t\right\}$$

$$= e^t\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t\right\}$$

$$= e^t\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$x^2(t) = e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

which is second linearly independent sol^n

Now,

Eigenvector corresponding to eigenvalue $\lambda = 2$

$$[A - \lambda I]v = 0.$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -v_1 + v_2 = 0$$
$$-v_2 = 0$$

$$\Rightarrow \quad v_1 = 0$$
$$v_2 = 0.$$

$\therefore$ v_3 is arbitrary.

$$\therefore \quad x^3(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is third L.I. solution.

The complete sol^n of the problem is

$$x = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$